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Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 68 (2004) 923–931

THE INTERACTION OF AN INFINITE WHEEL-TRAIN WITH A CONSTANT SPACING BETWEEN THE WHEELS MOVING UNIFORMLY OVER A RAIL TRACK†

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(*Received* 11 *September* 2003)

A periodic model of rail track in the form of an infinite Timoshenko beam held by massive elastoviscous supports with constant spacing is considered. The sprung part of the carriage is separated from the wheel by an elastic spring, and its action on the wheel in a first approximation is therefore represented by a static load. The steady vertical vibrations of the rail under the action of an infinite train of wheels moving uniformly over the rail are investigated. The distances between the wheels are identical, and all the wheels have the same mass and carry the same load. The static stiffness of the track over a sleeper exceeds that of track in the space between sleepers. Therefore, under the action of a constant load, each wheel performs vertical parametric vibrations with the frequency of passage of the sleepers. These vibrations are an extension of parametric vibrations described using the well-known Mathieu and Hill equations.

1. FORMULATION OF THE PROBLEM

Consider the motion of an infinite train of wheels with spacing h along a rail track with a sleeper spacing l. Each wheel has a mass m_0, carries a constant load a_0 and moves without detachment from the rail at a constant non-zero speed υ_0. Between the wheel and the rail an elastic interaction occurs which is modelled using a linear contact spring with stiffness k_c. The longitudinal coordinate will be denoted by x and the time by t. Let the point $x = 0$ coincide with one of the sleepers (Fig. 1), and the time $t = 0$ correspond to the passing of the "zeroth" wheel over the point $x = 0$.

The force with which the wheel acts on the rail has a period l/υ_0. A positive direction of this force corresponds to stretching of the contact spring. The forces acting on the wheel and on the rail are opposite to one another (Fig. 1). We will represent the periodic force with which the zeroth wheel acts on the rail in the form of a Fourier series

$$f_0(t) = -a_0 \sum_{m=-\infty}^{+\infty} F_m \exp\left(\frac{i2\pi m \upsilon_0 t}{l}\right), \quad f_0\left(t + \frac{l}{\upsilon_0}\right) = f_0(t) \tag{1.1}$$

Each term of the series has a period l/υ_0. The dimensionless Fourier coefficients F_m are to be determined.

The vertical deflection of the zeroth wheel is defined by the differential equation

$$m_0 \frac{d^2 y_0(t)}{dt^2} = -a_0 + a_0 \sum_{m=-\infty}^{+\infty} F_m \exp\left(\frac{i2\pi m \upsilon_0 t}{l}\right) \tag{1.2}$$

If $F_0 = 1$, this equation has a limited periodic solution.

The first wheel follows the zeroth wheel and proceeds to a fixed point of the rail with a delay of h/υ_0. The forces with which these two wheels act on the rail are related by the equality $f_1(t) = f_0(t - h/\upsilon_0)$.

†*Prikl. Mat. Mekh.* Vol. 68, No. 6, pp. 1025–1034, 2004.
0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.11.017

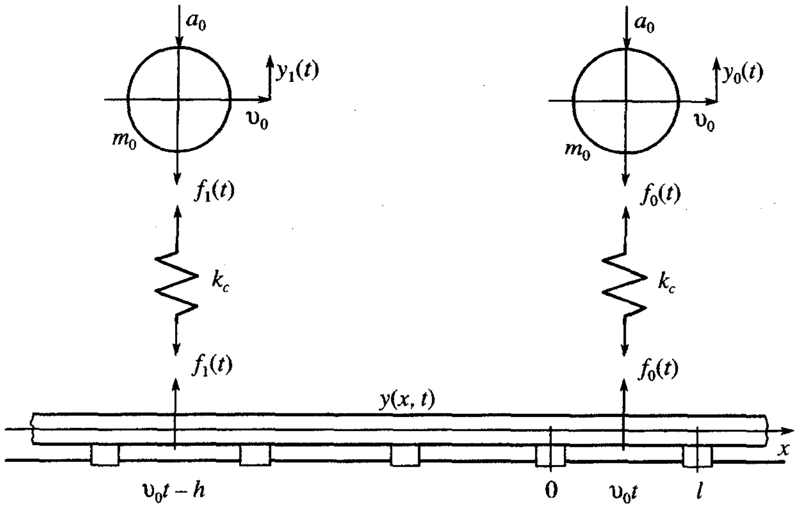

Fig. 1

The quantity $f_0(t - nh/v_0)$ is equal to the force of interaction of the rail and the wheel having the number n. Thus, the load on the rail from the infinite train of wheel is equal to

$$q(x, t) = \sum_{n=-\infty}^{+\infty} f_0(t - nh/v_0)\delta(x - v_0 t + nh) \tag{1.3}$$

The Dirac function $\delta(x - v_0 t + nh)$ specifies the concentrated force at the point $x_n = v_0 t - nh$ of contact of the rail and the wheel with number n.

The right-hand side of equality (1.3) does not change when t is replaced by $t + h/v_0$ and n is replaced by $n + 1$ simultaneously, nor when t is replaced by $t + l/v_0$ and x by $x + l$ simultaneously. Thus, the pressure of the wheels on the rail satisfies the following two periodicity conditions

$$q(x, t + h/v_0) = q(x, t), \quad q(x + l, t + l/v_0) = q(x, t) \tag{1.4}$$

2. THE STEADY VIBRATIONS OF THE RAIL TRACK

We will represent a sleeper as a concentrated mass held by a parallel spring and damper, and the rail as a Timoshenko beam of density per unit length ρ_0. The upward transverse deflection of the rail will be denoted by $y(x, t)$. The equation of the vertical vibrations of the rail has the form [1]

$$\begin{aligned} &EJ\frac{\partial^4 y(x,t)}{\partial x^4} + \rho_0\frac{\partial^2 y(x,t)}{\partial t^2} - \rho_0\left(\frac{J}{A} + \frac{EJ}{R}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} + \frac{\rho_0^2 J}{RA}\frac{\partial^4 y(x,t)}{\partial t^4} = \\ &= q(x,t) + \frac{\rho_0 J}{RA}\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{EJ}{R}\frac{\partial^2 q(x,t)}{\partial x^2} \end{aligned} \tag{2.1}$$

where EJ is the flexural stiffness of the rail, E is modulus of elasticity and J is the moment of inertia of the cross-section of the rail. The quantity $R = k'GA$ is called the shear stiffness, G is the shear modulus, A is the rail cross-sectional area and the coefficient k' takes into account the non-uniformity of distribution of the shearing force over the cross-section. The right-hand side of Eq. (2.1) satisfies conditions (1.4). Consequently, the left-hand side and the solution of the equation likewise satisfy these conditions. Thus, the transverse deflection of the rail satisfies two periodicity conditions

$$y(x, t + h/v_0) = y(x, t), \quad y(x + l, t + l/v_0) = y(x, t) \tag{2.2}$$

The first condition can be explained by the fact that identical wheels, having the same speed and carrying an identical load, pass over an arbitrary point of the rail x in equal time intervals h/υ_0. Hence, we can confine ourselves to the time interval $0 \le t \le h/\upsilon_0$. The second condition is a special case of the condition of steady vibrations of the rail that was examined in detail in [1]. If we assume that $h \ge l$, then, in the above-mentioned time interval, on the rail segment $0 \le x \le l$, bounded by two neighbouring sleepers, only the zeroth wheel will be there. Consequently, on the right-hand side of formula (1.3) we need only retain one non-zero term $f_0(t)\delta(x - \upsilon_0 t)$ corresponding to $n = 0$. According to formula (1.1), it is possible to adopt, as the load applied to a section of rail, the quantity

$$q(x, t) = -\sum_{m=-\infty}^{+\infty} F_m q_m(x, t), \quad q_m(x, t) = a_0 \exp\left(\frac{i2\pi m \upsilon_0 t}{l}\right)\delta(x - \upsilon_0 t) \tag{2.3}$$

In view of the linearity of the problem, we will represent the transverse deflection of the rail in the form

$$y(x, t) = -\sum_{n=-\infty}^{+\infty} F_n y_n(x, t) \tag{2.4}$$

where $y_n(x, t)$ is the transverse deflection of the rail under the action of the load $q_n(x, t)$ that satisfies the two conditions (2.2). The quantities $y_n(x, t)$ and $q_n(x, t)$ satisfy Eq. (2.1).

The partial derivatives $\partial y(x, t)/\partial x$ and $\partial^3 y(x, t)/\partial x^3$ have discontinuities at the points where the rail rests on sleepers. Therefore, the second condition of (2.2) leads to a boundary-value problem on the section $0 \le x \le l$ with the following boundary conditions:

$$\begin{gathered}\frac{\partial^j y(l, t + l/\upsilon_0)}{\partial x^j} = \frac{\partial^j y(0, t)}{\partial x^j} + \kappa_j k(t), \quad j = 0, 1, 2, 3 \\ k(t) = \rho_1 l \frac{\partial^2 y(0, t)}{\partial t^2} + rl\frac{\partial y(0, t)}{\partial t} + uly(0, t), \quad \kappa_0 = 0, \quad \kappa_1 = -R^{-1}, \quad \kappa_2 = 0, \quad \kappa_3 = (EJ)^{-1}\end{gathered} \tag{2.5}$$

The values of the partial derivatives mentioned earlier are taken at the point $x = 0$ to the right, and at the point $x = l$ to the left. The quantities $\rho_1 l$, rl and ul are the sleeper mass, the damper viscosity and the spring stiffness, expressed in terms of the parameters of the corresponding homogeneous elastoviscous base. The solution of the boundary-value problem, analogous to the problem investigated in [1], is extended to arbitrary values of x and t using periodicity conditions (2.2). The boundary conditions (2.5) have no meaning when $\upsilon_0 = 0$.

We will change to dimensionless time $T = \upsilon_0 t/l$ and dimensionless longitudinal coordinate $X = x/l$. Note that the dimensionless longitudinal coordinate of the zeroth wheel $X_0 = x_0/l = \upsilon_0 t/l$ coincides with T. We will assume that

$$\delta(l(X - T)) = \delta(X - T)/l, \quad \delta(X - T) = \delta(T - X)$$

The dimensionless quantity $Y_n(X, T) = y_n(x, t)/l$ satisfies the equation

$$\frac{\partial^4 Y_n(X, T)}{\partial X^4} + \alpha\frac{\partial^2 Y_n(X, T)}{\partial T^2} - (\beta + \gamma)\frac{\partial^4 Y_n(X, T)}{\partial X^2 \partial T^2} + \beta\gamma\frac{\partial^4 Y_n(X, T)}{\partial T^4} =$$

$$= A_0\left(1 + \psi\left(\beta\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2}\right)\right)\exp(i2\pi nT)\delta(X - T), \quad 0 \le X \le 1 \tag{2.6}$$

$$\alpha = \frac{\rho_0 \upsilon_0^2 l^2}{EJ}, \quad \beta = \frac{\rho_0 \upsilon_0^2}{EA}, \quad \gamma = \frac{\rho_0 \upsilon_0^2}{R}, \quad A_0 = \frac{a_0 l^2}{EJ}, \quad \psi = \frac{\gamma}{\alpha}$$

the two periodicity conditions

$$Y_n(X, T + H) = Y_n(X, T), \quad H = h/l, \quad Y_n(X + 1, T + 1) = Y_n(X, T) \tag{2.7}$$

and the four boundary conditions

$$\frac{\partial^j Y_n(1, T+1)}{\partial X^j} = \frac{\partial^j Y_n(0, T)}{\partial X^j} + \mathrm{K}_j K(T), \quad \mathrm{K}_0 = 0, \quad \mathrm{K}_1 = -\psi, \quad \mathrm{K}_2 = 0, \quad K_3 = 1$$

$$j = 0, 1, 2, 3; \quad K(T) = K_2 \frac{\partial^2 Y_n(0, T)}{\partial T^2} + K_1 \frac{\partial Y_n(0, T)}{\partial T} + K_0 Y_n(0, T) \tag{2.8}$$

$$K_0 = \frac{ul^4}{EJ}, \quad K_1 = \frac{r v_0 l^3}{EJ}, \quad K_2 = \frac{\rho_1 v_0^2 l^2}{EJ}$$

According to the first periodicity condition (2.7), the quantity $Y_n(X, T)$ can be represented in the form of a Fourier series

$$Y_n(X, T) = \sum_{s=-\infty}^{+\infty} C_{sn}(X) \exp(\mathrm{i}\Phi_s T); \quad \Phi_s = \frac{2\pi s}{H} \tag{2.9}$$

with unknown coefficient $C_{sn}(X)$, which satisfy the equality

$$C_{sn}(X) = \frac{1}{H}\int_0^H Y_n(X, T)\exp(-\mathrm{i}\Phi_s T)\mathrm{d}T \tag{2.10}$$

3. SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR $s \neq 0$

Let us calculate the Fourier coefficients $C_{sn}(X)$. For this, we multiply Eq. (2.6), the four boundary conditions (2.8) and also the second periodicity condition of (2.7) by the quantity $(\exp(-\mathrm{i}\Phi_s T)\mathrm{d}T/H$ and integrate from zero to H. Note that integration of the first condition of (2.7) leads to an identity. Integrating by parts and taking into account equality (2.10), we obtain the ordinary differential equation

$$\frac{\mathrm{d}^4 C_{sn}(X)}{\mathrm{d}X^4} + (\mathrm{B} + \Gamma)\frac{\mathrm{d}^2 C_{sn}(X)}{\mathrm{d}X^2} + (\mathrm{B}\Gamma - \mathrm{A})C_{sn}(X) = \frac{A_0}{H}(1 + \psi(\Phi_{sn}^2 - \mathrm{B}))\exp(-\mathrm{i}\Phi_{sn}X)$$

$$\Phi_{sn} = \Phi_s - 2\pi n, \quad \mathrm{A} = \alpha\Phi_s^2, \quad \mathrm{B} = \beta\Phi_s^2, \quad \Gamma = \gamma\Phi_s^2$$

The four boundary conditions

$$\exp(\mathrm{i}\Phi_s)\frac{\mathrm{d}^j C_{sn}(1)}{\mathrm{d}X^j} = \frac{\mathrm{d}^j C_{sn}(0)}{\mathrm{d}X^j} + \mathrm{K}_j K_*(\Phi_s)C_{sn}(0), \quad j = 0, 1, 2, 3$$
$$K_*(\Phi_s) = -\Phi_s^2 K_2 + \mathrm{i}\Phi_s K_1 + K_0 \tag{3.1}$$

define, on the section $0 \le X \le 1$, the boundary-value problem for this equation.
Integration of the second condition of (2.7) leads to the equality

$$\exp(\mathrm{i}\Phi_s)C_{sn}(X+1) = C_{sn}(X) \tag{3.2}$$

If the integer variables s is zero, then $\Phi_s = 0$, and on the left-hand side of the ordinary differential equation only the first term is retained. First, we will solve this problem for the condition that $s \neq 0$. In this case

$$C_{sn}(X) = \frac{A_0}{H}(\exp(-\mathrm{i}\Phi_{sn}X) - J(\Phi_s)N(X, \Phi_s))P_{sn}, \quad P_{sn} = \frac{1 + \psi(\Phi_{sn}^2 - \mathrm{B})}{L_{sn}}$$

$$2(\sigma_1^2+\sigma_2^2)N(X,\Phi_s) = \left((1-\psi\sigma_2^2)\frac{\mathrm{sh}(\sigma_1(1-X))+\exp(-\mathrm{i}\Phi_s)\mathrm{sh}(\sigma_1 X)}{\sigma_1(\cos\Phi_s-\mathrm{ch}\sigma_1)}-\right.$$

$$\left.-(1+\psi\sigma_1^2)\frac{\sin(\sigma_2(1-X))+\exp(-\mathrm{i}\Phi_s)\sin(\sigma_2 X)}{\sigma_2(\cos\Phi_s-\cos\sigma_2)}\right),\quad J(\Phi_s) = \frac{1}{1/K_*(\Phi_s)+N(0,\Phi_s)}$$

$$2\sigma_{2,1}^2 = ((\mathrm{B}-\Gamma)^2-4\mathrm{A})^{1/2}\pm(\mathrm{B}+\Gamma),\quad L_{sn} = \Phi_{sn}^4-(\mathrm{B}+\Gamma)\Phi_{sn}^2+\mathrm{B}\Gamma-\mathrm{A}$$

On the section $0 \le X \le 1$, we have the following expansion in a Fourier series

$$\exp(\mathrm{i}\Phi_s N)N(X,\Phi_s) = \sum_{m=-\infty}^{+\infty} Q_{sm}\exp(\mathrm{i}2\pi mX),\quad Q_{sm} = \frac{1+\psi((\Phi_{sm}^2-\mathrm{B}-\Gamma))}{L_{sm}}$$

using which we reduce the solution of the boundary-value problem to the form

$$C_{sn}(X) = \frac{A_0}{H}\left(\exp(-\mathrm{i}\Phi_{sn}X)-J(\Phi_s)\sum_{m=-\infty}^{+\infty}Q_{sm}\exp(-\mathrm{i}\Phi_{sm}X)\right)P_{sn} \tag{3.3}$$

The right-hand side of Eq. (3.3) satisfies condition (3.2). Consequently, the solution of the boundary-value problem in the form (3.3) can be used for any values of X.

4. SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR $s = 0$

When $s = 0$, the solution of (3.3) no longer has any meaning, while the ordinary differential equation, boundary conditions (3.1) and equality (3.2) acquire the form

$$\frac{\mathrm{d}^4 C_{0n}(X)}{\mathrm{d}X^4} = \frac{A_0}{H}(1+\psi(2\pi n)^2)\exp(\mathrm{i}2\pi nX),\quad 0\le X\le 1$$

$$\frac{\mathrm{d}^j C_{0n}(1)}{\mathrm{d}X^j} = \frac{\mathrm{d}^j C_{0n}(0)}{\mathrm{d}X^j}+\mathrm{K}_j K_0 C_{0n}(0),\quad j = 0,1,2,3;\quad C_{0n}(X+1) = C_{0n}(X) \tag{4.1}$$

$$C_{0n}(X) = \frac{A_0}{H}P_{0n}(\exp(\mathrm{i}2\pi nX)-1),\quad P_{0n} = \frac{1+\psi(2\pi n)^2}{(2\pi n)^4},\quad n\ne 0$$

$$C_{00}(X) = \frac{A_0}{H}\left(\frac{1}{K_0}+\frac{\psi}{2}X+\frac{1-12\psi}{24}X^2-\frac{1}{12}X^3+\frac{1}{24}X^4\right)$$

The coefficient $C_{0n}(X)$ is a periodic function. Consequently, the expression obtained for it holds for any value of X. The coefficient $C_{00}(X)$ takes equal values at the ends of the section, which is consistent with the boundary conditions. Expanding this coefficient in a Fourier series, we obtain its representation in the form of a periodic function

$$C_{00}(X) = \frac{A_0}{H}\left(\frac{1}{K_0}+\frac{\psi}{12}+\frac{1}{720}-\sum_{m\ne 0}P_{0m}\exp(\mathrm{i}2\pi mX)\right) \tag{4.2}$$

which again holds for any value of X.

5. TRANSVERSE DEFLECTION OF THE RAIL

We substitute expression (3.3), expression $C_{0n}(X)$ from (4.1) and expression (4.2) into equality (2.9). Changing the order of summation, we obtain

$$Y_n(X,T) = \frac{A_0}{H}\sum_{m=-\infty}^{+\infty}\exp(\mathrm{i}2\pi mX)W(m,n,T-X) \tag{5.1}$$

$$W(m,n,T-X) = S(m,n,T-X), \quad m \neq n, \quad m \neq 0, \quad n \neq 0$$
$$W(n,n,T-X) = S(n,n,T-X) + P_{0n}, \quad W(0,n,T-X) = S(0,n,T-X) - P_{0n}, \quad n \neq 0$$
$$W(m,0,T-X) = S(m,0,T-X) - P_{0m}, \quad m \neq 0$$
$$W(0,0,T-X) = \frac{1}{K_0} + \frac{\psi}{12} + \frac{1}{720} + S(0,0,T-X)$$
$$S(m,n,T-X) = \sum_{s \neq 0} (\Psi(m,n) - J(\Phi_s)Q_{sm})P_{sn}\exp(i\Phi_s(T-X)), \quad \Psi(m,n) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

The quantity $W(m, n, T-X)$ is represented by a series and can be calculated approximately by replacing the series $S(m, n, T-X)$ by the sum of a finite number of terms.

We divide both sides of equality (2.4) by l. We then replace $Y_n(X, T)$ using equality (5.1). As a result, we obtain the dimensionless transverse deflection of the rail

$$Y(X,T) = \frac{y(x,t)}{l} = -\frac{A_0}{H}\sum_{m=-\infty}^{+\infty}\exp(i2\pi mX)\sum_{n=-\infty}^{+\infty}F_n W(m,n,T-X) \tag{5.2}$$

In equality (5.2), the variables X and T are arbitrary numbers. Replacing X by T, we obtain the dimensionless transverse deflection of the rail at the point of contact with the zeroth wheel

$$Y(T,T) = -\frac{A_0}{H}\sum_{m=-\infty}^{+\infty}\exp(i2\pi mT)\sum_{n=-\infty}^{+\infty}F_n W(m,n,0) \tag{5.3}$$

6. INTERACTION OF THE WHEEL AND RAIL

Integrating Eq. (1.2) twice and changing to dimensionless variables, as a result of integration and formula (1.1) we obtain the dimensionless vertical deflection of the zeroth wheel and the dimensionless force of interaction of this wheel and the rail

$$Y_0(T) = \frac{y_0(t)}{l} = -A_0\sum_{m \neq 0}\frac{F_m\exp(i2\pi mT)}{M_0(2\pi m)^2}, \quad M_0 = \frac{m_0 v_0^2 l}{EJ} \tag{6.1}$$

$$F_0(T) = \frac{f_0(t)l^2}{EJ} = -A_0\sum_{m=-\infty}^{+\infty}F_m\exp(i2\pi mT) \tag{6.2}$$

A contact spring connects the wheel and the rail. The strain of the spring is equal to the quantity $f_0(t)/k_c$. The difference between the vertical deflections of the zeroth wheel and the rail at the point of its contact with the wheel is equal to this quantity. The corresponding dimensionless quantities, represented in the form of Fourier series (5.3), (6.1) and (6.2), are related by the equality

$$Y_0(T) = Y(T,T) + F_0(T)/K_c, \quad K_c = k_c l^3/(EJ) \tag{6.3}$$

Equality (6.3) holds if the coefficients of $\exp(i2\pi mT)$, where $m \neq 0$, are equal on its left- and right-hand sides. Taking into account also the equality $F_0 = 1$, we obtain the infinite system of equations

$$\left(\frac{1}{K_c} - \frac{1}{M_0(2\pi m)^2}\right)F_m + \frac{1}{H}\sum_{n \neq 0}F_n W(m,n,0) = -\frac{W(m,0,0)}{H}, \quad m \neq 0 \tag{6.4}$$

for determining the infinite number of unknown coefficients F_m. The only method of solving the infinite system of equations (6.4) is to approximate it by a finite system of equations. Taking into account that $1 \leq |m| \leq N$ and $1 \leq |n| \leq N$, we obtain a system of $2N$ equations containing $2N$ unknown coefficients. Solving the finite system of equations, we calculate these coefficients. Increasing N and assessing the

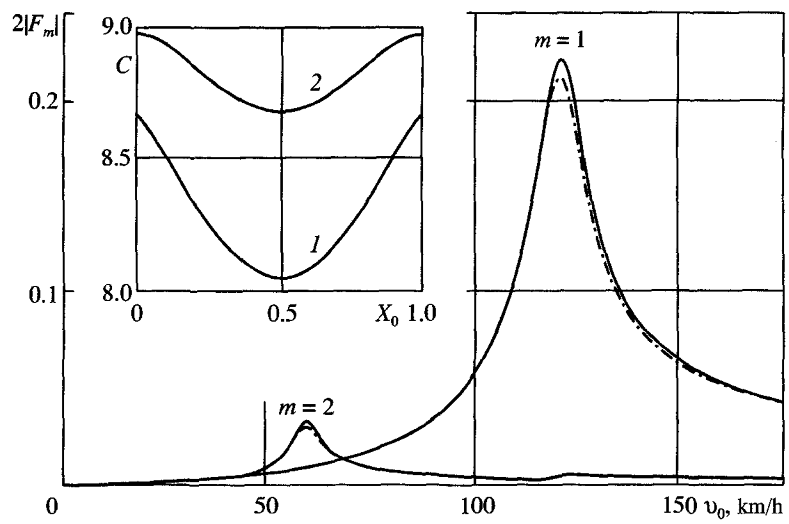

Fig. 2

effect of this increase on the coefficients already calculated, we can achieve the required accuracy. We then assume that $N = 5$.

The coefficients F_m and F_{-m} are complex-conjugate quantities. The real quantity $F_m\exp(12\pi mT)$ + $F_{-m}\exp(-\mathrm{i}2m/T)$ determines the harmonic component of the force of interaction of the wheel and the rail m, which has the dimensionless amplitude $2|F_m|$. Figure 2 shows the results of a calculation of this amplitude ($m = 1, 2$) as a function of the speed υ_0 for the following parameters of the rail track [2]:

$$EJ = 3.57\times 10^6\ \mathrm{N\ m^2},\quad A = 0.006\ \mathrm{m^2},\quad \rho_0 = 48\ \mathrm{kg/m},\quad k' = 0.34,\quad l = 0.8\ \mathrm{m}$$
$$u = 40\times 10^6\ \mathrm{N/m^2},\quad \rho_1 = 43.6\ \mathrm{kg/m},\quad r = 26\times 10^3\ \mathrm{N\ s/m^2}$$

and a wheel mass $m_0 = 700$ kg with a wheel spacing h equal to five sleeper spacings l (the solid curves) and with $h = 10l$ (the dashed curves). The difference between the solid and the dashed curves is a consequence of dynamic interaction of neighbouring wheels. The moving wheel generates travelling waves in the rail, which act on the neighbouring wheels. These waves attenuate due to damping of the sleepers. Transfer to the case $h = 15l$ leads to a barely appreciable change in the curves. When the wheel spacing is increased further, the curves hardly change. Thus, which $h = 15l$, the interaction of the wheels is negligible.

In the problem examined, the dimensionless quantity $H = h/l$ is an arbitrary positive number. If $H \to \infty$, the series given in the previous section are converted into the corresponding improper integrals obtained earlier [1] using the integral Fourier transformation and describing the motion of a single wheel over the rail track. For a fairly high value of H, these series serve as an approximation of the improper integrals, which shortens the calculations several fold. In subsequent calculations we will assume that $H \geq 20$ and consider the motion of a single wheel.

For a numerical investigation of the rail track dynamics, we will consider the finite section of the track, the ends of which are fastened by some method. The waves which arise when the wheel and track interact are reflected from the ends of the section of track. The reflected waves distort the results of a numerical investigation. Replacing the fastening of the ends of the track section by the condition of periodicity, Ripke [3] examined two cases of such conditions:

$$\pm y(x, t + h/\upsilon_0) = y(x, t)$$

In the first case, the boundary conditions at the start of the section and the boundary conditions at the end of the section are assumed to be identical. Thus, a track section whose length is a multiple of the sleeper spacing was replaced by a ring or "squirrel wheel". The motion of one wheel in this ring is equivalent to the previously examined motion of an infinite wheel-train with a spacing between wheels equal to the length of the ring along an infinite track. Travelling waves act on the wheel. A numerical investigation showed that this effect is negligible if the length of the ring $h \geq 15l$, which is consistent with the investigation given above. In the second case the boundary conditions at the ends of the track section were assumed to be opposite in sign. In this case, the track is represented as being laid on the well-known Möbius surface.

7. THE CHANGE IN THE STATIC STIFFNESS OF THE TRACK. THE CONNECTION WITH THE MATHIEU AND HILL EQUATIONS. THE STABILITY OF MOTION

In equality (5.3) we will replace the dimensionless time T by the dimensionless coordinate of the wheel $X_0 = \upsilon_0 t/l$. We will fix X_0. Let $\upsilon_0 \to 0$. Then the dimensionless parameter $M_0 \to 0$, and the time $t = X_0 l/\upsilon_0 \to \infty$. Furthermore, the quantities A, B, and Γ approach zero, while L_{sm}, $K_*(\Phi_s)$ and $N(0, \Phi_s)$ respectively are converted into Φ_{sm}^4, K_0 and

$$\frac{1-6\psi}{12(\cos\Phi_s - 1)} + \frac{1}{4(\cos\Phi_s - 1)^2}$$

If the wheel spacing h and the sleeper spacing l are incommensurable, the quantity Φ_{sm}^4 does not vanish, which makes it possible to use the formulae given above. By taking the limit, we establish that $F_m = 0$ for all non-zero values of m. Furthermore, $F_0 = 1$. Thus, equality (5.3) is converted into the following equality

$$Y(X_0, X_0) = -\frac{A_0}{H}\sum_{m=-\infty}^{+\infty} \exp(i2\pi m X_0) W(m, 0, 0) \tag{7.1}$$

and the periodic quantity

$$C(X_0) = -A_0/Y(X_0, X_0) \tag{7.2}$$

is the dimensionless track stiffness at the point X_0.

The left-hand part of Fig. 2 shows the change in the dimensionless track stiffness $C(X_0)$ in the period $0 \le X \le 1$. Curve 1 corresponds to the results of a calculation taking into account the shear strain in the rail. The greatest value of $C(X_0)$, equal to 8.669, is reached at points 0 and 1, which correspond to neighbouring sleepers. The lowest value of 8.050 at point 0.5 corresponds to the mid-span between sleepers. The change in stiffness, referred to its average value, is small (0.0741). Curve 2 is the result of a calculation ignoring the shear strain in the rail. The greatest value of 8.976 and the lowest value of 8.678 of the dimensionless track stiffness are reached at the same points. The absence of shear strain in the rail leads to an insignificant increase in the average track stiffness, which is largely determined by the flexural strain of the rail. Here, the relative change in stiffness, equal to 0.0338, is more than halved. Thus, the periodic change in rail track stiffness is largely connected with shear strain in the rail.

It is possible to arrive at the same result in a different way. We will retain the non-zero value of the speed of the wheel. We will ignore the mass of the wheel, rail and sleepers, and also the contact stiffness and viscous resistance of the track. Then the vertical deflections of the wheel and rail are equal, the quantities A, B, Γ and M_0 again vanish and consequently equalities (7.1) and (7.2) are retained. Thus, the transverse deflection of the rail is the same as its static deflection, i.e. the movement of the rail is quasi-static.

Once more, we will take into consideration the mass of the wheel and we will change, in equality (7.2), to dimensional quantities. Motion of the wheel over the rail leads to a periodic change in track stiffness at the point of contact. In the quasi-static approximation under examination, the vibrations of the wheel are similar to the vibrations of a concentrated mass loaded with a constant force and supported by a weightless spring with periodically changing stiffness. Thus, the vertical deflection of the wheel $y_0(t)$ is defined by an ordinary second-order linear differential equation

$$m_0\frac{d^2 y_0(t)}{dt^2} + \frac{EJ}{l^3}C\left(\frac{\upsilon_0 t}{l}\right)y_0(t) = -a_0 \tag{7.3}$$

with a constant right-hand side and a periodic coefficient at $y_0(t)$. If the periodic coefficient is expanded in a Fourier series, the left-hand side of Eq. (7.3) is identical with the left-hand side of Hill's equation [4, 5]. If, in the Fourier expansion, only the constant term and first harmonic are retained, the left-hand side of Eq. (7.3) will be identical with the left-hand side of Mathieu's equation [5, 6]. Note that the periodic coefficient can also be determined by representing the reactions of the sleepers using a series of Dirac functions with spacing l [7].

If the damping of vertical vibrations of the wheel is taken into account a first-order derivative appears in Eq. (7.3). With light damping, the solutions of the Hill and Mathieu equations may be unstable [4, 5] and increase without limit. The vertical vibrations of a solid wheel moving over rail track may also be unlimited. Note that the stability of the periodic model of a rail track has been investigated asymptotically [8. 9]. The motion of a mass along a beam lying on a periodically inhomogeneous viscoelastic base was considered in [9]. Numerical investigation of the monodromy matrix of differential equation (7.3) with damping confirmed the stability of the vibrations of the wheel for the track and wheel parameters given above [10]. The conclusions reached relate to the case where the unsprung part of the carriage is represented only by a single mass m_0. If account is taken of the bending of the axle of the wheel pair and one mass is replaced with two masses, related, respectively, to the wheel and axle box and connected to each other by a spring, then damping of one of the two masses representing the box may prove to be insufficient, and the entire system may be unstable.

Note the stability of the motion of an unloaded wheel along a rail supported by periodically positioned sleepers can be investigated by writing the second equality of (2.2) in the generalized form

$$y_n(x+l, t+l/v_0) = Ry_n(x, t)$$

where R is a dimensionless complex coefficient. In this case, the problem is reduced to the system of equations (6.4) with zero right-hand sides. The coefficients of this system depend on the assumed parameters of the track, on the mass and speed of the wheel and also on the dimensionless coefficient R. If the homogeneous system (6.4) has a non-zero solution provided that $|R| > 1$, then the quantity $y_n(x, t)$ increases exponentially, and the motion of the unloaded wheel turns out to be unstable. Note that, in the earlier investigation of the motion of an unbalanced wheel along a rail track [1], the analogous complex coefficient was equal to unity in absolute magnitude.

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Translated by P.S.C.